On Agent-Based Models of Mode Social Networks

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What is a (Social) Network?

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  - $V = \{v_1, v_2, \cdots, v_{|V|}\}$, $E = \{e_1, e_2, \cdots, e_{|E|}\}$.

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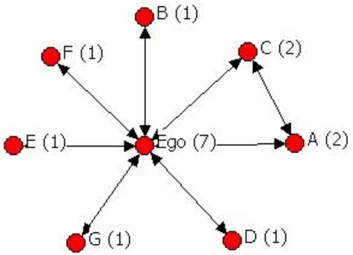
- Network Theory: Science of analyzing relations and linkages among actors within a network or among networks.
- Network Types:
    - One-mode vs Multi-mode Networks: One class of entities vs several classes of entities.
    - Stationary Networks vs Evolutionary Networks. Static vs Dynamic. Time-Series.
    - Binary (Dichotomous) vs Weighted (Valued).
    - Social vs other types.

Graph vs Matrix Representations

One-to-one correspondence between graphs and adjacency matrices.

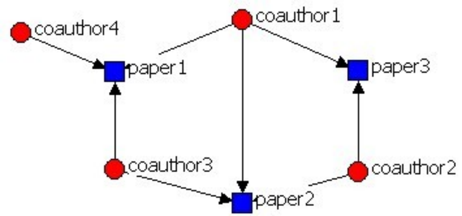


	Ego	A	B	C	D	E	F	G
Ego	0	1	1	1	1	1	1	1
A	1	0	0	1	0	0	0	0
B	1	0	0	0	0	0	0	0
C	1	1	0	0	0	0	0	0
D	1	0	0	0	0	0	0	0
E	1	0	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	1	0	0	0	0	0	0	0

One-Mode and Two-Mode Networks

- One-mode: One set of vertices.
- Two-mode (or also known as bipartite): Two sets of vertices.

Example (Co-authorship Networks)



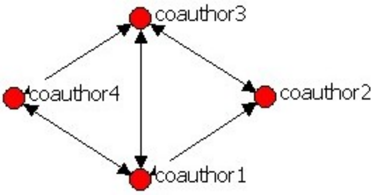


Figure: Left: Two-mode author-by-paper social network. Right: One-mode author-by-author social network.

One-Mode and Two-Mode Networks

Example (Continued)

The author-by-paper adjacency matrix *AP* for the previous example is

	paper1	paper2	paper3
coauthor1	1	0	1
coauthor2	0	1	1
coauthor3	1	1	0
coauthor4	1	0	0

$$\Rightarrow AP = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{4\times 3}$$

The author-by-author adjacency matrix *AA* for the previous example is

	author1	author2	author3	author4
author1	0	1	1	1
author2	1	0	1	0
author3	1	1	0	1
author4	1	0	1	0

$$\Rightarrow AA = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Mathematical Model

Let AP be the adjacency matrix of size $m \times n$ representing the graph of the network, with $m =$ number of coauthors, and $n =$ number of papers. Then,

$$AA_{m\times m} = AP_{m\times n} \cdot AP^T_{n\times m} = \text{ coauthorship adjacency matrix, and}$$

$$PP_{n\times n} = AP^T_{n\times m} \cdot AP_{m\times n} = \text{ paper-by-paper adjacency matrix.}$$

where,

$$aa_{ii} = \sum_{j=1}^{n} ap_{ij} = \text{number of papers author } i \text{ published,}$$

$$pp_{jj} = \sum_{i=1}^{m} a_{ij} = \text{number of coauthors coauthored paper } j\text{, and}$$

$$aa_{ij} = \text{edge weight (tie-strength) between coauthors } i \text{ and } j.$$

If $D_{m\times m} = AA^2_{m\times m}$ then $d_{ii} =$ vertex degree of coauthor i.

Formal Model

Definition

Consider the bipartite graph $G\left(V^a, V^b, E\right)$ with sets of vertices $V^a = \left\{v_1^a, v_2^a, \cdots, v_i^a, \cdots v_{|V^a|}^a\right\}$ of type A; $|V^a| = n$, $V^b = \left\{v_1^b, v_2^b, \cdots, v_j^b, \cdots v_{|V^b|}^b\right\}$ of type B; $|V^b| = m$, and a set of edges $E = \left\{e_1, e_2, \cdots, e_{|E|}\right\}$ connecting vertices of types A and B respectively.

The adjacency matrix $AB_{n\times m}$, also known as Edmonds matrix, is defined by

$$ {}^{w}AB_{n\times m} = \begin{cases} ab_{ij}, & \left(v_i^a, v_j^b\right) \in E \\ 0, & \left(v_i^a, v_j^b\right) \notin E. \end{cases} $$

$1 \le i \le n, \quad 1 \le j \le m$, and the indeterminate $ab_{ij} \in \mathbb{R}$.
If AB is binary then $ab_{ij} = 1$.

Notation: ${}^{w}AB$, ${}^{b}AB$.

Methodology

- We can mathematically reduce a two-mode network to a one-mode network.

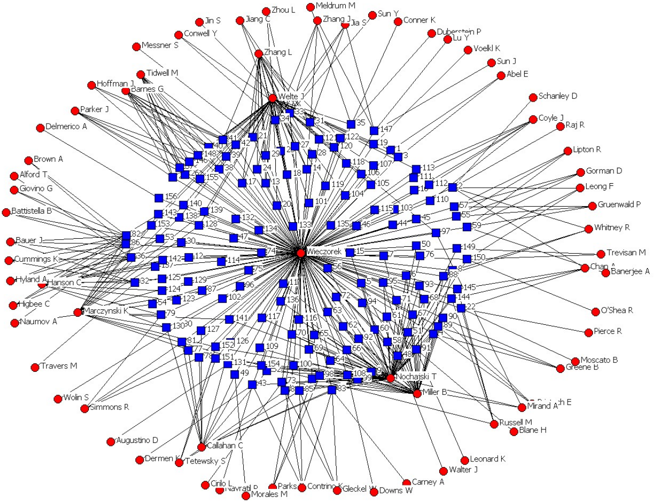
Relational Networks

Let ${}^{b}AB$ be a two-mode **binary** relational matrix of size $n \times m$ with respect to types A and B respectively, then

$${}^{w}AA_{n\times n} = {}^{b}AB_{n\times m} \cdot {}^{b}AB^{T}_{n\times m}$$

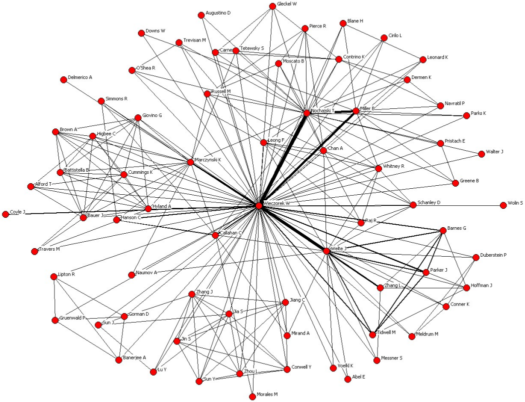
is the one-mode weighted matrix of vertices of type A related through vertices of type B.

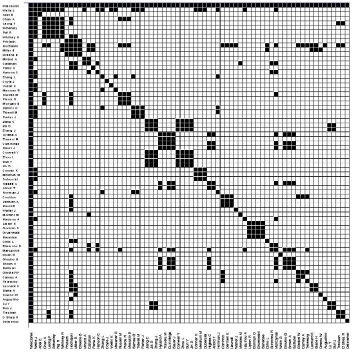
Two-mode Coauthorship Social Network



One-mode Coauthorship Social Network

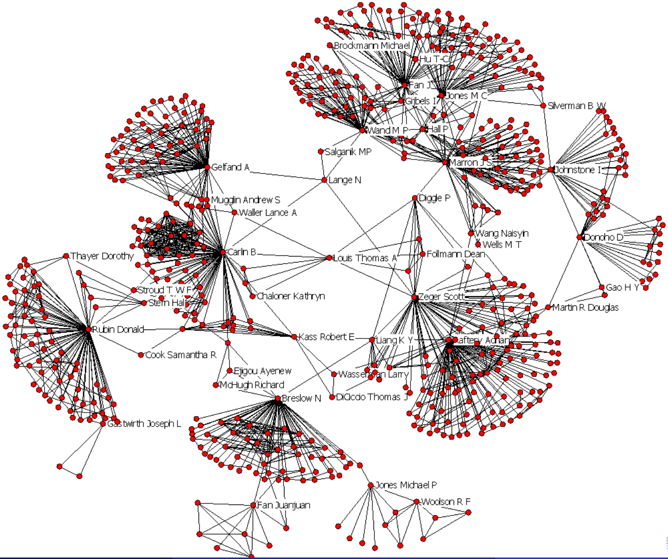
$$^{w}AA =^{b} AP \times^{b} AP^{T}$$



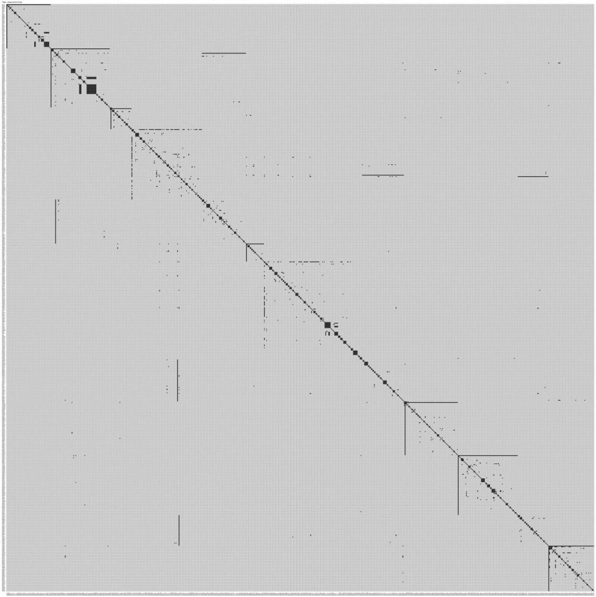


Another Coauthorship Social Networks

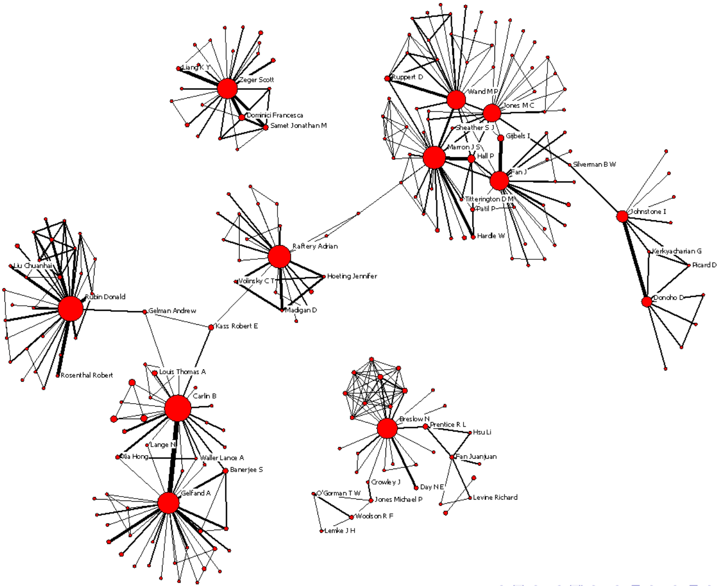
Source: CIS database. 1767 published papers and 874 unique authors.



Coauthorship Social Networks



Coauthorship Social Networks



Modeling Tripartite Networks

Definition

Let $V^a = \left\{v_1^a, v_2^a, \cdots, v_i^a, \cdots, v_{|V^a|}^a\right\}$ be the set of vertices of type a, $V^b = \left\{v_1^b, v_2^b, \cdots, v_j^b, \cdots, v_{|V^b|}^b\right\}$ be the set of vertices of type b, and $V^c = \left\{v_1^c, v_2^c, \cdots, v_k^c, \cdots, v_{|V^c|}^c\right\}$ be the set of vertices of type c.
Furthermore, let $E = \left\{e_1, e_2, \cdots, e_{|E|}\right\}$ be the set of edges connecting types a, b, c vertices; e_i is a hyperedge. Assume $|V^a| = n$, $|V^b| = m$, and $|V^c| = p$.
The binary adjacency matrix ${}^bABC_{n\times m\times p}$ corresponding to the finite graph $G\left(V^a, V^b, V^c, E\right)$ is defined by

$$abc_{ijk} = \begin{cases} 1, & \left(v_i^a, v_j^b, v_k^c\right) \in E \\ 0, & \text{otherwise.} \end{cases}$$

$1 \leq i \leq n, \quad 1 \leq j \leq m, \quad 1 \leq k \leq p.$

Manipulating Three-Mode Networks

- A cuboid is a three-dimensional object.
- More network features are revealed.
- There are several matrix arithmetic operations to perform on a cuboid matrix.
- Some result in a rectangular matrix; others result in a cuboid matrix.

Example

- A network of Author-by-Paper-by-Institution.
- A network of Alcohol outlets-by-Zip code-by-Income.

2D Projection

Consider the tripartite dichotomous matrix ${}^{b}ABC_{n\times m\times p}$.

${}^{b}ABC_{n\times m\times p}$ is a tensor of rank 3; I will use the term cuboid instead.

The marginal bipartite weighted matrix ${}^{w}AB_{n\times m}$ is

$$ {}^{w}ab_{ij} = \sum_{k=1}^{p} abc_{ijk}, \quad 1 \leq k \leq p. $$

- This is equivalent to projecting the 3D cuboid matrix onto the 2D planar matrix.
- The result is the marginal bipartite distribution for vertices of types A and B.

3D Transpose

- Unlike a 2D rectangular matrix (has only two faces), a 3D cuboid has six faces.
- Six different ways to view the block in terms of size; namely, $n \times m \times p$, $n \times p \times m$, $m \times n \times p$, $m \times p \times n$, $p \times n \times m$, and $p \times m \times n$.
- 3D transpose can be performed in six different ways.

Example (3D Transpose)

$$ABC^{T_{cba}}_{n\times m\times p} = CBA_{p\times m\times n}, \text{ with } cba_{ijk} = abc_{kji},$$

and

$$ABC^{T_{bac}}_{n\times m\times p} = BAC_{m\times n\times p}, \text{ with } bac_{ijk} = abc_{jik}.$$

3-D Two-Mode Network

Traditional 3D matrix product giving another 3D matrix.

$$^{w}AAC_{n\times n\times p} = {}^{b}ABC_{n\times m\times p} \cdot {}^{b}ABC^{T_{bac}}_{n\times m\times p},$$

$AB_{n\times m} \cdot BA_{m\times n}$ must be well-defined,

$$^{w}aac_{k} = {}^{b}ABC_{k} \cdot {}^{b}BAC_{k}, \quad 1 \leq k \leq p.$$

$^{w}AAC_{n\times n\times p}$ is a two-mode graph (network) represented by a 3D matrix.

3-D One-Mode Network

Given ${}^{w}AAC_{n\times n\times p}$.

$$ {}^{w}AAA_{n\times n\times n} = {}^{w}AAC_{n\times n\times p} \cdot {}^{w}AAC^{T_{caa}}_{n\times n\times p}, $$

$AC_{n\times p} \cdot CA_{p\times n}$ must be well-defined,

$$ {}^{w}aaa_{i} = {}^{w}AAC_{i} \cdot {}^{b}CAA_{i} \ . $$

- ${}^{w}AAA_{n\times n\times n}$ is the one-mode 3D matrix of triadic vertices (triplets) of type A related through vertices of types B and C.
- ${}^{w}AAA_{n\times n\times n}$ contains hyperedges connecting three vertices.

The Hyper Product $A \circ B$

Hyper Cuboid

$$^{w}ABBA_{n\times m\times m\times n} = {}^{b}ABC_{n\times m\times p} \circ {}^{b}ABC^{T_{cba}}_{n\times m\times p},$$

$BC_{m\times p} \cdot CB_{p\times m}$ must be defined.

$$^{w}abba_{kl} = ^{b} ABC_{kl} \cdot ^{b} CBA_{kl}, \quad 1 \leq k, l \leq p.$$

- $^{w}ABBA_{n\times m\times m\times n}$ is the hyper-cuboid two-mode adjacency matrix of pair of vertices (v_i, v_j) of types A and B related through the set of vertices v_k of type C.
- The 4D hyper-matrix can be represented using 3D matrix by stacking n cuboid matrices each of size $m \times m \times n$, resulting in a 3D matrix of size $m \times m \times n^2$.

From Three-Mode to One-Mode in One Step

2-D One-Mode Network

$$^{w}CC_{p\times p} = {}^{b}ABC_{n\times m\times p} \odot {}^{b}ABC^{T_{bac}}_{n\times m\times p},$$

$ABC_{n\times m} \cdot BAC_{m\times n}$ must be well-defined,

$$^{w}cc_{ij} = \sum_{q=1}^{n}\sum_{r=1}^{n} {}^{b}ABC_{(n\times m)_i} \cdot {}^{b}BAC_{(m\times n)_j}$$

$$= \sum_{q=1}^{n}\sum_{r=1}^{n} {}^{w}AA_{(qr)_{ij}}, \quad 1 \leq i,j \leq p.$$

- $^{w}CC_{p\times p}$ is the one-mode 2D matrix of pairwise vertices of type C related through vertices of types A and B.
- $^{w}CC_{p\times p}$ represents diadic relations.

Generalized $N-$Mode Model

- $N-$hyper cuboid matrix (tensor of rank N).
- Multidimensional Networks.
- Use hyper graphs (cliques).
- Lower mode relations may be retrieved in the same fashion we convert three-mode to two-mode and two-mode to one-mode networks.

Generalized $N-$Mode Model

Definition

Let $V^1 = \left\{v_1^1, v_2^1, \cdots, v_{i_1}^1, \cdots, v_{|V^1|}^1\right\}$, $V^2 = \left\{v_1^2, v_2^2, \cdots, v_{i_2}^2, \cdots, v_{|V^2|}^2\right\}$, $V^3 = \left\{v_1^3, v_2^3, \cdots, v_{i_3}^3, \cdots, v_{|V^3|}^3\right\}, \cdots,$ $V^N = \left\{v_1^N, v_2^N, \cdots, v_{i_N}^N, \cdots, v_{|V^N|}^N\right\}$ be the sets of vertices of type $1, 2, 3, \cdots, N$ respectively. Furthermore, let $E = \left\{e_1, e_2, \cdots, e_{|E|}\right\}$ be the set of edges connecting types $1, 2, 3, \cdots, N$ vertices. Once again, e_i is a hyperedge. Assume $|V^i| = n_i, \forall\ 1 \le i \le N$.

The binary adjacency matrix $A_{n_1 \times n_2 \times \cdots \times n_N}$ for the finite graph $G(V^1, V^2, \cdots, V^N, E)$ is defined by

$$a_{i_1 i_2 \cdots i_N} = \begin{cases} 1, & \left(v_{i_1}^1, v_{i_2}^2, \cdots, v_{i_N}^N\right) \in E \\ 0, & \text{otherwise.} \end{cases}$$

$1 \le i_j \le n_j$, for $1 \le j \le N$.

Reducing the $N-$mode network to a two-mode network

Consider the hyper-matrix $A_{n_1 \times n_2 \times \cdots \times n_N}$.

Then, ${}^{w}A_{n_i \times n_j}$, $1 \leq i, j \leq N$, $i \neq j$, is the two-mode matrix where,

$$a_{ij} = \sum_{k_1=1}^{n_{k_1}} \cdots \sum_{k_{N-2}=1}^{n_{k_{N-2}}} a_{i_1 i_2 \cdots i_N}, \quad 1 \leq k_l \leq n_l.$$

This is equivalent to projecting from $N-$dimensional hyper-cuboid matrix onto the 2D planar matrix giving the marginal $N-$mode distribution for types i and j.

$N-$D Transpose

Suppose $A_{n_1 \times n_2 \times \cdots \times n_N}$ is a binary $N-$partite matrix. Then,

$$A^{T_{m_1 \times m_2 \times \cdots \times m_N}}_{n_1 \times n_2 \times \cdots \times n_N} = A_{m_1 \times m_2 \times \cdots \times m_N}, \quad a_{i_1 i_2 \cdots i_N} = a_{j_1 j_2 \cdots j_N},$$

where $j_1 j_2 \cdots j_N$ is a permutation of $i_1 i_2 \cdots i_N$.

Reducing the $N-$mode network to a one-mode network

Let ${}^{b}A_{n_1 \times n_2 \times \cdots \times n_N}$ be a binary $N-$mode matrix. Assume $m_1 = n_i$, the $(N-1)-$mode matrix for a mode i is

$$\begin{aligned} {}^{w}A_{m_1 \times m_1 \times m_3 \times \cdots \times m_N} &= {}^{b}A_{m_1 \times m_2 \times m_3 \times \cdots \times m_N} \cdot {}^{b}A^{T_{m_2 m_1 m_3 \cdots m_N}}_{m_1 \times m_2 \times m_3 \times \cdots \times m_N} \\ &= {}^{b}A_{m_1 \times m_2 \times m_3 \times \cdots \times m_N} \cdot {}^{b}A_{m_2 \times m_1 \times m_3 \times \cdots \times m_N}, \end{aligned}$$

The product of the sub-matrices $A_{m_1 \times m_2} \cdot A_{m_2 \times m_1}$ must be well-defined.

Thus, m_2 mode is being eliminated.

The one-mode matrix ${}^{w}A_{m_1 \times m_1 \times \cdots \times m_1}$ for a mode i is obtained in $N-1$ matrix multiplications and transpose.

${}^{w}A_{n_i \times n_i \times \cdots \times n_i} = {}^{w}A_{m_1 \times m_1 \times \cdots \times m_1}$ is the one-mode $N-$dimensional matrix of $N-$cliques of type i related through all other types.

Hyperedge connecting $N-$vertices.

Problem!

The traditional method of multiplying the matrix by its transpose fails to give meaningful one-mode edge weights if the two-mode matrix is not binary; i.e. edges have real values.

Applications

- Online Friendship Social Networks.
  - Facebook, Livejournal, MySpace.
- Coauthorship Social Networks.
  - "Who-Wrote-With-Whom".
- Covert/Espionage (Alliances) Social Networks.
  - "Who-Talked-With-Whom"
- Disease Propagation Social Networks.
- Computer Networks.
- Term-Document Networks.
- Protein Networks.

Current Research

- Developing mathematics underpinning networks
- Infinite Networks and Multidimensional Networks.
- Mechanism to store networks using matrices.
- Estimating diadic relations: Edges and Vertices.
- Estimating triadic relations: Hyperedges.
- Expressing weighted matrices in terms of binary matrices.
- Multi-Mode Networks.
- Modeling the generalized $N-$mode networks on real data.
- Estimate missing edges and vertices.
- Studying relationships between cliques and hypergraphs.
- Evolutionary networks
- Preferential attachment.
- Stochastic models.
- Text mining.

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Shameless Propaganda

Look for the Third International Conference on Social Computing, Behavioral Modeling and Prediction
SBP(2010)

To be held in Washington, DC
March-April 2010

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